STABILITY OF AN INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS

By

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Qualitative properties (boundedness, stability, etc.) of solutions of dynamical systems, described by infinite-systems of differential equations, are discussed. Explicit results are obtained for a special class of systems, denoted by (S), which describe the process of dissociation of polymer molecules into smaller units. This process is customarily described by a finite system of ordinary differential equations. But, we consider the "limit case" of an infinite system of such equations to understand the limit behavior of very large systems.

The positivity property of (S) is proved: if every component of the initial condition is nonnegative, then the solution of (S) remains nonnegative componentwise. The purpose of this dissertation is to investigate the stability property of (S). This is done by using a Liapunov function in the framework of Hale [1]. The result of the

investigation covers a general case which goes beyond the physical interpretation of (S). That is, solutions of (S) have the property of stability in ℓ_1. The same result is also derived by a direct argument using Barbalat's Lemma. Finally, in the appendix we discuss the approximation of the solutions for (S).

V. M. Popov

Chairman

CHAPTER I

INTRODUCTION

This dissertation contains an investigation of some qualitative properties (boundedness, stability, etc.) of the solutions of dynamical systems, described by infinite systems of differential equations. Such problems have been studied by many authors who have obtained basic general results (R. Bellman [1], J. Hale [1], etc.).

Our main purpose is to obtain explicit results for a special class of such systems, introduced and studied by Oguztörelli [1], R. Bellman [1], etc. These systems describe the process of dissociation of polymer molecules into smaller units (i.e. the degradation of polymers). This process, like its opposite, polymerization, are customarily described by finite systems of ordinary differential equations. Following Oguztörelli [1], we consider the "limit-case" of an infinite system of such equations. It is hoped that this approach would lead to a better understanding of the limit behavior of very large systems. Oguztörelli's system is described by the differential equations

$$(S)\qquad \begin{aligned} \dot{x}_1 &= \sum_{j>1} m_{1j}x_j \\ \dot{x}_i &= -\alpha_i x_i + \sum_{j>i} m_{ij}x_j, \quad i = 2,3,\ldots \end{aligned}$$

with the initial condition $x_i(0) = c_i$, $i = 1,2,\ldots$. Here the α_i's, the m_{ij}'s and the c_i's are nonnegative constants. One assumes that

$$\sum_{i=1}^{\infty} c_i < \infty .$$

Thus the system can be written in the form $\dot{x} = Ax$ where the infinite matrix of the system has the form:

$$A = \begin{pmatrix} 0 & m_{12} & m_{13} & \\ & -\alpha_2 & m_{23} & \ldots\ldots \\ 0 & & -\alpha_3 & \end{pmatrix}$$

One assumes that

1. $0 < \alpha_j < M$, $i = 1,2,\ldots$, $j = 2,3,\ldots$ for some positive number M.

2. $$\alpha_j = \sum_{i=1}^{j-1} m_{ij}$$

Guided by the physical interpretation of the equations, we choose to study this system in the space ℓ_1. Then A becomes a continuous linear operator from ℓ_1, into ℓ_1 (Taylor [1], p. 183), and therefore the system has a unique solution $x(t) = e^{At}x_0$ (Ladas and

Lakshmikantham [1] p. 24 or Dieudonne [1], p. 289).

Chapter III is devoted to proving that "if every component of the initial condition is positive, the solution of the system (S) remains positive componentwise." This property is suggested by the physical interpretation and is very easy to prove in the finite-dimensional case. However in our infinite-dimensional case the problem is much more delicate and we need the special study from Chapter III to solve it. From the point of view of the theory of dynamical systems this result means that the set $P := \{x \in \ell_1 : x_i > 0, i = 1,2,\ldots\}$ is positively invariant with respect to our system. The set $K := \{x \in \ell_1 : x_i \geq 0, i = 1,2,\ldots\}$ is also positively invariant. An interesting feature is that these sets are nowhere dense in ℓ_1, and yet these are the only subsets of ℓ_1 which are involved in the interpretation of the problem. The fact that in our application we only need to consider the solutions which belong to a nowhere dense subset of our Banach space seems to suggest that the Banach space is "too large" for some applications and that - at least for some applications - it is of primary importance to develop a special theory of dynamical systems on meager sets in Banach spaces. Our results belong to this direction of research. The concepts of dynamical systems and stability in subsets of a Banach space appear in LaSalle [1], Lefschetz [1] and Hale [1]. Since these ideas are basic for our needs we give a systematic presentation in Chapter II.

In Chapter V our results are based on the work of

J. Hale [1] about dynamical systems in Banach spaces and the theory of ω-limit sets. We use especially his procedure of proving that a bounded orbit has in some cases a nonempty ω-limit set by introducing the concept of a limit dynamical system: let u be a dynamical system on Banach spaces B and C where $B \subset C$. If an orbit belongs to a bounded set in B and a compact set in C, then the orbit has a nonempty ω-limit set which is invariant. We prove that an orbit of our system (S) in ℓ_1 has a nonempty ω-limit set in ℓ_∞. We do this by showing that the orbit belongs to a cube Q of the form

$$Q = \{x \varepsilon \ell_\infty : \ |x_i| \le s_i \text{ and } s_i \to 0 \text{ as } i \to \infty\}$$

and Q is a compact set in ℓ_∞. Our next results are based on the work of LaSalle [1], especially on his invariance principle and theorems of J. Hale in [1]. With these tools we find Liapunov functions (in the extended sense from the quoted work) and obtain supplementary information about the qualitative behavior of the solution of our system. It is important to mention that this extension of the Liapunov theory is essential to our needs. The stationary solution of our system is not asymptotically stable in the classical sense, but it is asymptotically stable in the extended sense and also in the practical sense.

The theme of Chapter VI is to prove that "the solution of the system (S) starting from the initial condition x_0 in P approaches the stationary point $(\|x_0\|, 0,0,\ldots)$." This is

done by a direct argument, without applying any standard stability theory. The main tool we use in this Chapter is Barbalat's lemma.

In the Appendix we discuss the problem of approximating the solution of our infinite-dimensional system by the solutions of a suitable sequence of finite dimensional systems (the "truncated systems"). We find sufficient conditions for the validity of such a procedure (the so-called "principe des Reduites," as studied in R. Bellman [1]).

CHAPTER II

DEFINITIONS AND CONCEPTS OF STABILITY THEORY

In this chapter we discuss some basic concepts of stability theory. The purpose of re-examining these concepts and definitions is to cover the case when stability concepts are defined only on a subset of the Banach space under consideration. This prepares the road for considering the stability properties of the system (S) given in the Introduction. For system (S) we are led to discuss the stability property on a meager set of the Banach space ℓ_1, because the whole space is too large for our consideration.

§ 2.1 Motions and Dynamical Systems

We first recall a few basic concepts of the theory of stability. We follow closely the approach of W. Hahn [1, § 35] with some simplifications, allowed by our framework.

Let B and X be Banach spaces, x an element in X with its norm $||x||_X$, b an element in B with its norm $||b||_B$. The zero element of X or B will be called the origin.

We consider elements p in X which depend on three parameters; t, b and t_0. The parameter t is assumed to satisfy the condition $t_0 \leq t < \infty$. The parameter b ranges over a subset G of the space B. The element p will be

interpreted as a point or as a map, depending on the context. Let $R_{t_0} = \{t \in R: t \geq t_0\}$ and p be a map from $R_{t_0} \times G \times R$ into X such that

$$p = p(t,b,t_0)$$

is continuous.

Definition 1

For a fixed b, the function

$$p_b: R_{t_0} \times \{t_0\} \to X \tag{1.1}$$

defined as $p_b(t,t_0) = p(t,b,t_0)$ is called the motion determined by b and t_0.

Definition 2

If $p(t + s, b, t_0 + s) = p(t,b,t_0)$ for every $s \geq 0$, the motion is called autonomous. In this case, without loss of generality we assume $t_0 = 0$ and simply write $p(t,b)$ and $p_b(t)$ instead of $p(t,b,t_0)$ and $p_b(t,t_0)$.

Examples of Motions

1. Solutions of a system of ordinary differential equations in Euclidean space.

2. Solutions of functional differential equations with finite (or infinite) retardation (Ref. Example 2 of Hale [1]).

3. Solutions of differential equations on abstract spaces (Ladas and Lakshmikantham [1]).

As the parameter b ranges over G, (1.1) defines a family of motions. Let $p(t,b,t_0)$ and $p(t,b',t_0)$ be two

different motions of the family belonging to two different parameter values b and b', $b \neq b'$. Then the norm

$$||p_b(t,t_0) - p_{b'}(t,t_0)||_X$$

can be interpreted as the distance of the two motions at time t. We call the motion p_b the undisturbed motion. The parameter b' is supposed to belong to a certain (relative) neighborhood of b in G, for instance

$$||b' - b||_B < r,\ b'\varepsilon G \text{ for some } r > 0.$$

We use the notation

$$B_G(b,\delta) = \{b'\varepsilon G:\ ||b' - b||_B < \delta\}.$$

A motion $p_{b'}$ is called a perturbed motion.

Definition 3

The unperturbed motion p_b is called stable with respect to G, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $b'\varepsilon B_G(b,\delta)$ one has the inequality $||p_b(t,t_0) - p_{b'}(t,t_0)||_X < \varepsilon$ for every $t \geq t_0$.

Definition 4

The unperturbed motion p_b is called attractive if for every $\eta > 0$ there exist a number T and a number $\delta > 0$ such that for every $b'\varepsilon B_G(b,\delta)$ and every $t > t_0 + T$ one has the inequality

$$||p_b(t,t_0) - p_{b'}(t,t_0)||_X < \eta.$$

B and X are not necessarily the same space in general though

they were usually so assumed (Hahn [1], § 35).

Suppose now that the spaces B and X are identical and that p $(t_0,b,t_0) = b$. Then b is called the initial point. We further assume that the motion $p(t,b',t_0)$ exist for all b' in $B_G(b,\delta)$ and for all $t \geq t_0$ and that the relationship

$$p(t,p(t_1,b,t_0),t_1) = p(t,b,t_0), t_0 \leq t_1 \leq t,$$

holds. If the motion is autonomous and B = X we make use of the following definitions concerning the function

$$p: R_+ \text{ X } G \rightarrow G \text{ (where G is a subset of X)}$$

Definition 5

We say that p defines a dynamical system on G if the mapping p satisfies the conditions

1. $p(0,b) = b$, for every $b \varepsilon G$,

2. $p(t_2,p(t_1,b)) = p(t_1+t_2,b)$, for every t_1, $t_2 \geq 0$ and every $b \varepsilon G$,

3. p is continuous.

Definition 6

A set $M \subset G$ is called positively invariant if $a \varepsilon M$ implies $p(t,b) \varepsilon M$ for all $t \geq 0$.

Definition 7

The set $\gamma^+(b)$ defined by

$$\gamma^+(b) = \bigcup_{t \geq 0} p(t,b),$$

is called the (positive) orbit of the motion starting from b at t = 0.

§ 2.2 Invariant Sets

Hale [1] defined the concept of an invariant set as follows:

Definition 8

A set M in X is an invariant set of the dynamical system p if for each ψ in M there is a function $U: R \times M \to M$ such that $1°\cdot U(0,\psi) = \psi$, for every ψ in M, and 2°, for any σ in R, $p(t,U(\sigma,\psi)) = U(t+\sigma, \psi)$ for every t in R_+.

Remark. Since $U(t,\psi) = p(t,U(0,\psi)) = p(t,\psi)$ for all $t \geq 0$, the function U can be considered as an extension of p to the negative real numbers. Since $p(t,U(-t,\psi)) = U(0,\psi) = \psi$, it follows that $U(-t,\psi)$ is a solution, in M, of the equation (with unknown ψ) $p(t,\psi) = \psi$. Evidently a set which is invariant in the sense of Hale (Definition 8) is positively invariant, but not vice versa.

§ 2.3 Positively Invariant Sets

We introduce some new concepts related to invariant sets:

Definition 9

Let $A \subset X$. The set $\tilde{A}$ which is the union of all positive orbits starting in A is called the post-set of A. That is,

$$\tilde{A} = \bigcup_{a \in A} \gamma^+(a)$$

Suppose A is an invariant set. Any subset B in A such that $\tilde{B} = A$ is called a pre-set of A. In particular $\tilde{A} = A$. A few lemmas follow naturally from the definitions above.

Lemma 2.1

The postset $\tilde{A}$ is positively invariant for any set $A \subset X$.

Proof: If x is in $\tilde{A}$, then there is an element a in A such that $x \varepsilon \gamma^+(a)$. Thus,

$$\gamma^+(x) \subset \tilde{A}.$$

Lemma 2.2

A X is a positively invariant set if $A = \tilde{A}$.

Proof: Clear.

Lemma 2.3

If $B \subset A$ and $\tilde{B} = A$, then either $B = A$ or B is not positively invariant.

Proof: Suppose B is positively invariant, then $B = \tilde{B}$ by Lemma 2.2. Since $\tilde{B} = A$ by the hypothesis, $B = \tilde{B} = A$.

Remark. If the system is unstable in A, then there is a proper preset of A. But if A consists of equilibrium points or the system is asymptotically stable on A, then there is no proper preset for A.

The following theorem gives a sufficient condition for a positively invariant set to become an invariant set (in the sense of Hale).

Theorem

Suppose M is a positively invariant set and ε is a positive number such that for every x in M there is y, $y \neq x$, in M so that $x = p(t,y)$ and $t > \varepsilon$. Then M is invariant.

Proof: Suppose ψ is a point in M. By the hypothesis there is a_1 in M, $a_1 \neq \psi$, and $s_1 > \varepsilon$ such that $\psi = p(s_1,a_1)$. Then there is a_2 in M, $a_2 \neq a_1$, and $s_2 > \varepsilon$ such that $a_1 = p(s_2,a_2)$ and so on. Therefore, there is a sequence $\{a_n\}$ from M and a sequence of positive numbers $\{t_n\}$, $t_n > n\varepsilon$, such that, for $n = 1,2, \ldots$,

$$a_k \varepsilon \gamma^+(a_n) \text{ if } k < n$$

$$\psi = p(t_n,a_n)$$

and

$$a_m = p(t_m - t_n,a_n), \; n < m.$$

Since $t_n \to \infty$ as $n \to \infty$, we can define a function U on M so that it satisfies the conditions for an invariant set (Definition 8) in the following way: $U(t,\psi) = p(t,\psi)$ if $t \geq 0$ and $U(t,\psi) = p(t_n + t,a_n)$ if $t < 0$ and $|t| \leq t_n$ for some t_n.

U is well-defined in the sense that if $|t| \leq t_n < t_m$, then

$$U(t,\psi) = p(t_m + t,a_m) = p(t_n + t,a_n)$$

because

$$p(t_m + t,a_m) = p(t_n + t + t_m - t_n,a_m)$$

$$= p(t_n + t,p(t_m - t_n,a_m)) = p(t_n + t,a_n).$$

Clearly,

$$U(0,\psi) = p(0,\psi) = \psi.$$

The next condition for invariance that we want to prove is: $p(t,U(\sigma,\psi)) = U(t + \sigma,\psi)$ for all t in R^+ and σ in R. If $\sigma \geq 0$, then,

$$U(\sigma,\psi) = p(\sigma,\psi)$$

and

$$\begin{aligned} p(t,U(\sigma,\psi)) &= p(t,p(\sigma,\psi)) \\ &= p(t + \sigma,\psi) = U(t + \sigma,\psi). \end{aligned}$$

Suppose $\sigma < 0$ and $|\sigma| \leq t_n$ for some n. Then $t_n + \sigma \geq 0$, $t + t_n + \sigma > 0$ and

$$\begin{aligned} p(t,U(\sigma,\psi)) &= p(t,p(t_n + \sigma,a_n)) \\ &= p(t + t_n + \sigma,a_n). \end{aligned}$$

If $t + \sigma \geq 0$, then,

$$\begin{aligned} p(t,U(\sigma,\psi)) &= p(t + t_n + \sigma,a_n) = p(t + \sigma,p(t_n,a_n)) \\ &= p(t + \sigma,\psi) = U(t + \sigma,\psi). \end{aligned}$$

In case $t + \sigma < 0$, since $t > 0$ and $\sigma < 0$ we have $|t + \sigma| < |\sigma| \leq |t_n|$. Therefore,

$$p(t,U(\sigma,\psi)) = p(t + \sigma + t_n,a_n) = U(t + \sigma,\psi)$$

by the definition of U. This implies that M is an

invariant set.

§ 2.4 Stability with Respect to a Subset

We now recall other definitions we need, following again W. Hahn [1].

Definition 11

Let p be a dynamical system on a subset G of X. For any x in G, the ω-limit set Ω_x of the orbit through x is the set defined as follows: $y \varepsilon \Omega_x$ if there exists an increasing sequency $\{t_n\}$, $t_n > 0$, $t_n \to \infty$ as $n \to \infty$ such that $||p(t_n,x) - y|| \to 0$ as $n \to \infty$. This definition is equivalent to

$$\Omega_x = \bigcap_{\tau \geq 0} C\ell \bigcup_{t \geq \tau} p(t,x)$$

Definition 12

The distance of a point x from a set M is defined by

$$d(x,M) = \inf\{||x - y||,\ y \varepsilon M.\}$$

If x is in M, $d(x,M) = 0$.

Definition 13

A positively invariant set M is called stable relative to G, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(b,M) < \delta$ and b in G imply the inequality

$$d(p(t,b),M) < \varepsilon.$$

Definition 14

A positively invariant set M is attractive relative

to G is

$$\lim_{t\to\infty} d(p(t,b),M) = 0$$

for every b in G.

Definition 15

A positively invariant set M is called asymptotically stable relative to G if it is stable and attractive relative to G.

Definition 16

The motion p(t,a) is called orbitally stable relative to G if the invariant set $\gamma^+(s)$ is stable relative to G. The motion is called orbitally attractive relative to G if $\gamma^+(a)$ is attractive relative to G. If both of these obtain simultaneously the motion is called orbitally asymptotically stable relative to G.

Definition 17

The motion p: $R_+ \times G \to G$ has the asymptotic equilibrium relative to G if every motion p(t,a), a in G, converges to a limit y in $\overline{G}$, the closure of G in X, as $t \to \infty$ and conversely to every element y in $\overline{G}$ there exists a point a in G such that

$$\lim_{t\to\infty} p(t,a) = y.$$

Definition 18

The unperturbed motion is called unstable if it is not stable. In this case there exists a number $\varepsilon > 0$, a

sequence $b_n \to a$ and a sequence $\{t_n\}$ such that

$$||p(t_n,a) - p(t_n,a_n)|| \geq \varepsilon$$

for every n.

CHAPTER III

POSITIVE SET AND POSITIVELY INVARIANT SETS

The positive cone K and the positive set P in ℓ_1 are defined, respectively, by

$$K := \{x \epsilon \ell_1 : \ x = (x_i) \text{ and } x_i \geq 0 \text{ for every } i\},$$

and

$$P := \{x \epsilon \ell_1 \ : x = (x_i) \text{ and } x_i > 0 \text{ for every } i\}.$$

The main purpose of this chapter is to show that the sets P and K, defined above, are positively invariant for the system (S).

A finite-dimensional system

$$\frac{dx}{dt} = Ax, \ x(o) = c$$

where $A = (a_{ij})$ is a constant matrix and c is an n-dimensional vector, has this property if and only if

$$a_{ij} \geq 0 \quad i \neq j \quad \text{(Bellman [1], p. 172).}$$

We shall show that the set

$$\tilde{P} = \{x \text{ in } \ell_1 : x = (x_i) \text{ and } x_i > 0 \text{ for } i \neq 1\}$$

is also a positively invariant set for the system (S).

Consider the function $\pi : R \to R^+$ defined by

$$\pi(\rho) = \begin{cases} \rho & \text{if } \rho \geq 0 \\ 0 & \text{if } \rho < 0 \,. \end{cases}$$

Then

$$\pi(\rho) = \frac{\rho + |\rho|}{2} \quad .$$

We recall that a function $f : R \to R$ is called subadditive if $f(\rho + \rho') \leq f(\rho) + f(\rho')$ for every ρ and ρ' in R.

<u>Lemma 3.1</u>

π is a subadditive function with

$$|\pi(\rho) - \pi(\rho')| \leq |\rho - \rho'|$$

for every ρ and ρ' in R.

Proof: Since

$$\begin{aligned} \pi(\rho + \rho') &= \frac{\rho + \rho' + |\rho + \rho'|}{2} \\ &\leq \frac{\rho + \rho' + |\rho| + |\rho'|}{2} \\ &= \frac{\rho + |\rho|}{2} + \frac{\rho' + |\rho'|}{2} \\ &= \pi(\rho) + \pi(\rho'), \end{aligned}$$

π is subadditive.

Also,

$$|\pi(\rho) - \pi(\rho') = |\frac{\rho + |\rho|}{2} - \frac{\rho' + |\rho'|}{2}|$$

$$= |\frac{\rho - \rho'}{2} + \frac{|\rho|-|\rho'|}{2}| \leq \frac{|\rho-\rho'|}{2} + \frac{||\rho| - |\rho'||}{2}$$

$$\leq |\rho - \rho'| .$$

Let π be the function from ℓ_1 into ℓ_1 defined by $\pi(x) = (\pi(x_1), \pi(x_2), \ldots)$ for every $x = (x_i)$ in ℓ_1.

Lemma 3.2

π is continuous, and if f is a continuous function ~~from~~ ℓ_1 into ℓ_1 which satisfies the Lipschitz condition, then $f \circ \pi$ also satisfies the Lipschitz condition.

Proof: By Lemma 2.1,

$$||\pi(x) - \pi(y)||_1 = \sum_i |\pi(x_i) - \pi(y_i)|$$

$$\leq \sum_i |x_i - y_i| = ||x - y||_1$$

for every x and y in ℓ_1. Therefore π is continuous. If $||f(x) - f(y)||_1 \leq K \; ||x - y||_1$ for some $K > 0$, and for every x,y in ℓ_1, then

$$||f(\pi(x)) - f(\pi(y))||_1 \leq K||\pi(x) - \pi(y)||_1 \leq K||x - y||_1.$$

Thus $f \circ \pi$ also satisfies a Lipschitz condition.

Remark. P and $K = \overline{P}$, the closure of P in ℓ_1, are meager sets of ℓ_1. But K_∞ and P_∞ are not meager sets in ℓ_∞, where

$$K_\infty = \{x \varepsilon \ell_\infty : x = (x_i) \text{ and } x_i \geq 0 \text{ for every } i\}$$

and

$$P_\infty = \{x \varepsilon \ell_\infty : x = (x_i) \text{ and } x_i > 0 \text{ for every } i\}.$$

Lemma 3.3

Suppose (S) and (T) are systems defined in ℓ_1 such that

$$\text{(S)} \quad \dot{x} = f(x), \quad x(0) = x_0, \quad t \geq 0$$

has a unique solution $x(t,x_0)$, and

$$\text{(T)} \quad \dot{\tilde{x}} = g(\tilde{x}), \quad \tilde{x}(0) = x_0, \quad t \geq 0$$

has a solution $x(t,x_0)$. Suppose $f(y) = g(y)$ whenever $y = \tilde{x}(t,x_0)$ for some $t \geq 0$; that is, the positive orbit of $\tilde{x}(t,x_0)$ is contained in the set $\{y \varepsilon \ell_1 : f(y) = g(y)\}$. Then $\tilde{x}(t,x_0)$ is the solution of (S).

Proof: Suppose $\tilde{x}(t,x_0)$ is a solution of (T), then $g(\tilde{x}(t,x_0)) = f(\tilde{x}(t,x_0))$ for all $t \geq 0$ by the hypothesis. Thus $\tilde{x}(t,x_0)$ satisfies (S), and, since (S) has a unique solution, $\tilde{x}(t,x_0)$ is the solution of (S).

Let $f_i = \sum_{j>i} m_{ij} x_j$ and write the system (S) as

$$\text{(S)} \quad \begin{aligned} \dot{x}_1 &= f_1(x) \\ \dot{x}_i &= -\alpha_i x_i + f_i(x), \; i = 2,3,\ldots \; . \end{aligned}$$

The system (S) can be changed into a new system by the transformation

$$Y_i(t) = e^{\alpha_i t}\, x_i(t),\ i = 1,2,\ldots\ .$$

Then

$$\dot{Y}_i(t) = \alpha_i e^{\alpha_i t}\, x_i(t) + e^{\alpha_i t}\, \dot{x}_i(t)$$

$$= \alpha_i e^{\alpha_i t}\, x_i(t) + e^{\alpha_i t}\{-\alpha_i x_i(t) + f_i(x(t))\}$$

$$= e^{\alpha_i t}\, f_i(x(t)) = e^{\alpha_i t}\, f_i(\rho(y,t))$$

where $\rho(y,t) = (\rho_1(y,t), \rho_2(y,t), \ldots)$ and

$$\rho_i(y,t) = e^{-\alpha_i t}\, y_i(t).$$

This gives us a new system

$$(\tilde{S}) \quad \dot{Y}_i = g_i(t,y),\ y(0) = x_0,\ i = 1,2,3,\ldots .$$

where $g_i(t,y) = e^{\alpha_i t}\, f_i(\rho(y,t))$.

Next, we consider the system

$$(T) \quad \dot{z}_i = h_i(t,z),\ z(0) = x,\ i = 1,2,\ldots .$$

where $h_i(t,z) = g_i(t,\pi(z))$.

Consider the double sequence a_{ij}, $i,j, = 1,2,\ldots,$ defined by

$$a_{ij} = \begin{cases} m_{ij}\pi(x_j) & \text{if } j > i \\ 0 & \text{if } j \le i \end{cases} .$$

We assume, as in Chapter I (Assumption 2), that

$$\alpha_j = \sum_{i=1}^{j-1} m_{ij}.$$

Then, we have

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} m_{ij}\pi(x_j)$$

$$= \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} m_{ij}\ \pi(x_j) = \sum_{j=2}^{\infty} \alpha_j \pi(x_j) \qquad (3.1)$$

If the double sum is finite then, by Theorem 8.3 of Rudin [2],

$$\sum_{j=2}^{\infty} \sum_{i=1}^{j-1} m_{ij}\pi(x_j) = \sum_{i=1} \sum_{j=1} a_{ij}$$

$$= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} m_{ij}\pi(x_j) \qquad (3.2)$$

We need the following lemmas.

Lemma 3.4

Suppose that $0 < \alpha_i < M$ for every i and that x in ℓ_1. Then

$$\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} m_{ij}\pi(x_j) \leq M\ ||x||_1.$$

Proof: Since

$$\sum_{j=2}^{\infty} \alpha_j \pi(x_j) \leq M\ ||x||_1,$$

equalities (3.1) and (3.2) imply that the assertion of the lemma is true.

Lemma 3.5

If $a < b$ and $0 \leq t_1 < t_2$, then

$$|e^{bt_2} - e^{bt_1}| > |e^{at_2} - e^{at_1}| \quad .$$

Proof: Let $\beta = at_2 - at_1$. Then

$$e^{bt_1}(e^{\beta} - 1) > e^{at_1}(e^{\beta} - 1).$$

Since $\beta < bt_2 - bt_1$, we have

$$e^{bt_2} - e^{bt_1} > e^{bt_1}(e^{\beta} - 1)$$

$$> e^{at_1}(e^{\beta} - 1) = e^{at_2} - e^{at_1} \quad .$$

Lemma 3.6

If $r(t)$ satisfies the equation $\frac{dr}{dt} = b\cdot r\cdot \exp(ct)$ and $r(0) = r_0$, then $r(t) = r_0\exp\{\frac{b}{c}(e^{ct} - 1)\}$, $t \geq 0$.

Proof: Since $r'/r = be^{ct}$, we have

$$\int_0^t \frac{r'}{r}\, d\tau = \frac{b}{c}\int_0^t e^{c\tau} c\, d\tau$$

or

$$\ell n\, r(\tau)\Big|_0^t = \frac{b}{c}\, e^{c\tau}\Big|_0^t \quad .$$

This gives $\ell n r(t) - \ell n r_0 = \frac{b}{c}\,(e^{ct} - 1)$ or

$$\frac{r(t)}{r_0} = \exp\{\frac{b}{c}(e \quad - 1)\}\ .$$

We also need the following fact from Hille and Phillips [1] (Theorem 5.6.1).

<u>Theorem 3.7</u>

Let X be a Banach space. Let α, β, γ and μ be positive numbers such that $\alpha\mu \le \beta$. Let $f{:}RXX \to X$. Suppose that f is continuous in each variable separately for $|t - t_0| \le \alpha$, $||x - x_0|| \le \beta$ and that f satisfies the inequalities:

$$||f(t,x)|| \le \mu, \quad ||f(t,x_1) - f(t,x_2)|| \le \gamma\ ||x_1 - x_2||$$

for every t, x, x_1 and x_2 in the indicated regions. Then there exists one and only one strongly continuous differentiable function x(t) such that

$$x(t) = f[t, x(t)] \text{ in } |t - t_0| \le \alpha \text{ and } x(t_0) = x_0.$$

The proof of the following theorem is based on Theorem 3.7.

<u>Theorem 3.8</u>

Suppose that $0 < \alpha_i < M$ for every i. Then the system (T) has a unique solution locally through any point in $RX\ell_1$.

Proof: Let α and β be two positive numbers. Assume $|t - t_0| \le \alpha$ and $||z - z_0||_1 \le \beta$. For a fixed z in ℓ_1, one has

$$||h(t_1,z) - h(t\ z)||_1 = \sum_i |e^{\alpha_i t_1} f_i(\pi(\rho(z,t)))$$

$$- e^{\alpha_i t_2} f_i(\pi(\rho(z,t)))$$

$$= \sum_i |e^{\alpha_i t_1} \sum_j m_{ij} e^{-\alpha_j t_1} \pi(z_j) - e^{\alpha_i t_2} \sum_j m_{ij} e^{-\alpha_j t_2} \pi(z_j)|$$

$$\leq \sum_i \sum_j m_{ij} \pi(z_j) |e^{(\alpha_i - \alpha_j)t_1} - e^{(\alpha_i - \alpha_j)t_2}|.$$

Since $|\alpha_i - \alpha_j| < M$ for every i and j, Lemma 3.5 implies

$$|e^{(\alpha_i-\alpha_j)t_1} - e^{(\alpha_i-\alpha_j)t_2}| \leq |e^{Mt_1} - e^{Mt_2}|, \ (0 \leq t_1 < t_2).$$

Then we have (using Lemma 3.4)

$$||h(t_1,z) - h(t_2,z)||_1 \leq M \cdot ||z||_1 \cdot |e^{Mt_1} - e^{Mt_2}|.$$

Since e^{Mt} is continuous in t, h(t,z) is continuous in t.

Since

$$||h(t,z)||_1 = \sum_i |e^{\alpha_i t} \sum_j m_{ij} e^{-\alpha_j t} \pi(z_j)|$$

$$< e^{Mt}\ M\ ||z||_1,$$

there is a constant μ such that $||h(t,z)||_1 \leq \mu$ in the given region. Also,

$$||h(t,z) - h(t,\tilde{z})||_1 \leq \sum_i \sum_j e^{(\alpha_i - \alpha_j)t} m_{ij} |\pi(z_j) - \pi(\tilde{z}_j)|$$

$$< e^{Mt} \cdot M \cdot ||z - \tilde{z}||_1.$$

Therefore, there is a number γ such that

$$||h(t,z) - h(t,\tilde{z})||_1 \leq \gamma||z-\tilde{z}||_1, \quad |t-t_0| < \alpha.$$

If necessary, replace α by a smaller positive number so as to have $\alpha\mu \leq \beta$. (Observe that μ can be kept constant.) By Theorem 3.7 the system (T) has a unique local solution through (t_0,z_0).

Theorem 3.9

The system (T) has a solution $z(t)$ defined on R_+.

Remark. The following proof is based on Theorem 2.1 of Ladas and Lakshmikantham [2].

Proof: Since, as shown in the proof of Theorem 3.8, h is continuous in t and z separately and locally Lipschitz in z, then we have

$$||h(t_1,z) - h(t_2,\tilde{z})||_1 \leq ||h(t_1,z) - h(t_1,\tilde{z})||_1$$

$$+||h(t_1,\tilde{z}) - h(t_2,\tilde{z})||_1 \leq \gamma||z-\tilde{z}||_1+||h(t_1,\tilde{z}) - h(t_2,\tilde{z})||_1.$$

Therefore, h is continuous in both variables. Let $G(t,||z||_1) = Me^{Mt}||z||_1$, then, by the argument in the proof of Theorem 3.8, one obtains $||h(t,z)||_1 \leq G(t,||z||_1)$. Clearly, $G \in C[R_+ X \ell_1, R_+]$ and $G(t,r)$ is nondecreasing in $r \geq 0$, for each t in R_+.

By Lemma 3.6 the maximal solution of the scalar initial value problem $r' = Me^{Mt}r$ and $r(0) = r_0$ is $r(t,r_0) = r_0 \exp(e^{Mt}-1)$ which exists on R_+. Therefore, by Theorem 2.1 of Ladas and Lakshmikantham [2], the largest interval of existence of any solution of (T) with

$||z_0||_1 \leq r_0$ is R_+.

Corollary 3.10

The system ($\tilde{S}$) has a unique solution defined on R_+.

Proof: The argument is essentially the same as in Theorems 3.8 and 3.9 except that $\pi(z)$ is replaced by z. Notice that $||\pi(z)||_1 = ||z||_1$.

Theorem 3.11

P is an invariant set of (S).

Proof: Suppose z is a solution of the system (T), then $g_i(t,\pi(z)) = e^{\alpha it}f_i(\tilde{\pi}(z,t))$, where $\tilde{\pi}(z,t) = (\tilde{\pi}(z,t), \tilde{\pi}(z_2,t), \ldots)$ and $\tilde{\pi}(z_i,t) = e^{-\alpha it}\pi(z_i)$. Noting that $f_i(x) \geq 0$ for every i if x is an element from K, we have $h_i(t,z) \geq 0$ for $t \geq 0$ and every z in ℓ_1. Then, since every $h_i(t,z)$ is continuous in t,

$$z_i(t) = z_i(0) + \int_0^t h_i(\tau,z)d\tau > 0 \text{ if } z_i(0) > 0.$$

Since $\pi(z) = z$ if z is in P, we have $g_i(t,z) = h_i(t,z)$ if $z\varepsilon P$. Therefore, by Lemma 3.3, z(t) is the solution of ($\tilde{S}$). Since systems (S) and ($\tilde{S}$) have unique solutions, it follows that $x(t) = (e^{-\alpha it}z_i(t))$ is the solution of the system (S). Thus we have proved that if the initial condition $x(0) = x_0$ is in P, then the solution x(t) remains in P thereafter.

Corollary 3.12

The set $\tilde{P} = \{x\varepsilon\ell_1: x = (x_i), x_i > 0 \text{ for } i \neq 1\}$ is an invariant subset of the system (S).

Proof: Since $f_i(x) \geq 0$ for every i if $x = (x_i)$ is a

point in ℓ_1 such that $x_i > 0$ for $i = 2,3,\ldots$, we can repeat the argument in the proof of Theorem 3.11 with very little change.

Collary 3.13

The set K is an invariant subset of the system (S).

Proof: We can repeat the argument in the proof of Theorem 3.11 except that

$$z_i(t) = z_i(0) + \int_0^t h_i(\tau,z)d\tau \geq 0 \text{ if } z_i(0) \geq 0.$$

On the other hand, the mapping $x_0 \mapsto x(t,x_0)$ is continuous from ℓ_1 onto ℓ_1 for any $t \geq 0$, by Theorem (10.8.4) of Dieudonne [1]. Therefore:

Theorem 3.14

K is an invariant set of the system (S).

Proof: If $x_0 \varepsilon P$, then by the invariance of P, $x(t) \varepsilon P$ for $t \geq 0$. Now suppose $x_0 \varepsilon K-P$; this means there exists a sequence of points $\{P_n\}$ in P, and $\lim_{n\to\infty} P_n = x_0$. By the continuous dependence of the solution $x(t,x_0)$ on the initial condition x_0 mentioned above, we have

$$\lim_{n\to\infty} x(t,P_n) = x(t,x_0).$$

Since $\{x(t,P_n)\}$ P, $x(t,x_0) \varepsilon \overline{P} = K$, the closure of P in ℓ_1.

CHAPTER IV

c_0-SET AND THE ω-LIMIT SET

Let $f_i(x) = \sum_{j>i} m_{ij}x_j$ and write the system (S) as

$$\text{(S)} \qquad \begin{aligned} \dot{x}_1 &= f_1(x) \\ \dot{x}_i &= -\alpha_i x_i + f_i(x), \; i = 2,3,\ldots \end{aligned}$$

Then,

<u>Theorem 4.1</u>

If x_0 is in K and $x(t)$ is the solution of (S), then

$s(t) := \sum_{i=1}^{\infty} x_i(t)$ is constant for all $t \geq 0$.

Proof: By Theorem 3.14, $x(t) \geq 0$ for every $t \geq 0$. Since $m_{ij} \geq 0$, we have $f_i(x) \geq 0$ for every i, and by Theorem 12.21 of Hewitt and Stromberg [1], we have

$$\int \sum_i f_i \, dt = \sum_i \int f_i \, dt \tag{4.1}$$

and

$$\int \sum \alpha_i x_i \, dt = \sum_i \int \alpha_i x_i \, dt \tag{4.2}$$

Since $x_j \geq 0$, we have $\sum_{j>1} \alpha_j x_j \leq M \, ||x||_1$, where $M = \sup_{j>1} \alpha_j$.

Thus Theorem 8.3 of Rudin [1] gives

$$\sum_{i=1}^{\infty} f_i(x) = \sum_{i=2}^{\infty} \alpha_i x_i \tag{4.3}$$

By (4.1), (4.2) and (4.3)

$$\sum_{i=1}^{\infty} f_i \; dt = \sum_{i=2}^{\infty} \alpha_i x_i \; dt$$

or

$$f_i \; dt + \sum_{i=2}^{\infty} (f_i - \alpha_i x_i) dt = 0 \tag{4.4}$$

Applying this result to the system (S), one obtains

$$\sum_{i=1}^{\infty} \int_0^t \dot{x}_i(\tau) \; d\tau$$

$$= \int_0^t f_i(x(\tau) \; d\tau + \sum_{i=2}^{\infty} \int_0^t \{f_i(x(\tau)) - \alpha_i x_i(\tau)\} \; d\tau$$

$$= 0.$$

Since $\int_0^t \dot{x}_i(\tau) \; d\tau = x_i(t) - x_i(0)$,

$$0 = \sum_{i=1} \int_0^t \dot{x}_i(\tau) \; d\tau = \sum_{i=1} \{x_i(t) - x_i(0)\}$$

and thus $\quad \sum_{i=1}^{\infty} x_i(t) = \sum_{i=1}^{\infty} x_i(0)$.

<u>Remark</u>. The conclusion of Theorem 3.1 implies the stability of the system with respect to the set K.

Lemma 4.2

For every positive integer n and x_0 in K,

$s_n(t) := \sum_{i=n+1}^{\infty} x_i(t)$ is differentiable. Moreover, s_n is monotonically decreasing for every n.

Proof: Since $x_1(t) = x_2(t) + \ldots + x_n(t) + s_n(t)$ = constant for $t \geq 0$, by Theorem 4.1, we have

$$
\begin{aligned}
-\dot{s}_n(t) &= \dot{x}_1(t) + \dot{x}_2(t) + \ldots. + \dot{x}_n(t) \\
&= m_{12}x_2(t) + m_{13}x_3(t) + \ldots. \\
&- \alpha_2 x_2(t) + m_{23}x_3(t) + \ldots. \\
&\ldots\ldots\ldots\ldots\ldots \\
&= \sum_{j=n+1}^{\infty} \text{(nonnegative number) } x_j(t).
\end{aligned}
$$

Since $x(0) = x_0$ is in K, it follows that $x_i(t) \geq 0$ for every i by Theorem 3.14. Therefore, $\dot{s}_n(t) \leq 0$ for $t \geq 0$.

Let c_0 be the linear space of all sequences $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$. If X is a Banach space of sequences, for instance, ℓ_p, $1 \leq p \leq \infty$, we consider a special kind of subsets of X.

Definition

A set Q in a Banach space X of sequences is called a c_0-set if there is a sequence $\{a_n\}$ from c_0 such that, the condition $x = (x_n) \varepsilon Q$ is equivalent to $|x_n| \leq a_n$ for every n. If the last condition is replaced by $0 \leq x_n \leq a_n$ for every n, then we call Q a positive c_0-set.

This chapter is devoted to showing that every orbit of our system (S) starting in the positive set P belongs to a positive c_0-set in ℓ_1. Moreover, every c_0-set in ℓ_1 can be embedded in a compact set in ℓ_∞. Thus, we can discuss the ω-limit set of the system (S) embedded in ℓ_∞.

<u>Lemma 4.3</u>

The orbit of the system (S) starting in the positive set P belongs to a positive c_0-set in ℓ_1.

Proof: Let $x(t)$ be the solution of the system (S) starting from x_0 in P, $x_0 = x(0)$, $x_i(0) = c_i$ and $\sum c_i = c$. We want to show that there exists a sequence $\{a_n\}$ from c_0 such that, for every $t \geq 0$, $x_n(t) \leq a_n$ is true for every n. We consider the functions $s_n(t)$ introduced in Lemma 4.2. That is, $s_n(t) = \sum_{i>n} x_i(t)$ and $a_n = s_n(0) = \sum_{i>n} c_i$. Then $a_n \to 0$ as $n \to \infty$ and, for each $n = 1,2,\ldots, s_n(t)$ is monotonically decreasing by Lemma 4.2. Therefore, $0 < x_1(t) \leq c$ and $0 < x_n(t) \leq s_{n-1}(t) \leq a_{n-1}$ for $n = 2,3,\ldots,$ $t \geq 0$. This implies that the orbit $\gamma_0^+(x)$ of the solution $x(t)$ starting from x_0 in P belongs to a positive c_0-set.

<u>Remark</u>. By the natural injection $i: \ell_1 \to \ell_\infty$ defined by $i(x) = (x)$, the positive c_0-set in ℓ_1 can be considered as a proper subset of a positive c_0-set in ℓ_∞.

<u>Lemma 4.4</u>

A positive c_0-set Q^+ in ℓ_∞ is homeomorphic to the cartesian product $\sum_{i=1}^{\infty} I$ of countably many unit intervals.

Proof: Define

$$\phi: Q^{+} \rightarrow \prod_{i=1}^{\infty} I$$

by

$$\phi(x_1,x_2,.......) = (\frac{x_1}{a_1}, \frac{x_2}{a_2},......), \ a_1 = c.$$

Clearly ϕ is bijective. ϕ is continuous, since the projection on each factoer factor is continuous. To show that ϕ^{-1} is continuous we metrize $\prod_{i=1}^{\infty} I$ following Theorem 7.2 (2) Dugundji [1], p. 190: indeed, for $x = \{x_n\}$ and $y = \{y_n\}$, $d_n(x_n,y_n) = a_n|x_n - y_n|$, $n = 1,2,....$, defines a metric on the unit interval I, and we denote the metric space (I, d_n) by I_n. Then the diameter $S_n(I_n) = a_n$ and

$$\lim_{n\to\infty} S_n(I_n) = \lim_{n\to\infty} a_n = 0.$$

Therefore, ρ , defined by $\rho(x,y) = \sup\{d_n(x_n,y_n)\}=\sup\{s_n|x_n-y_n|\}$, metrizes the cartesian product topology of the space $\prod_{i=1}^{\infty} I$. If $\rho(x,y) < S$ in ΠI, then we have

$$||\phi^{-1}(x) - \phi^{-1}(y)||_{\infty} = \sup\{|s_n \ x \ n - s_n y_n|\}$$

$$= \sup\{s_n|x_n - y_n|\} = \rho(x,y) < S.$$

This completes the proof.

<u>Remark</u>. We closely followed the proof of Dugundji [1], p. 193.

Corollary 4.5

Q^+ is a compact subset in ℓ_∞.

Proof: By Lemma 4.2 and Tychonoff's Theorem in p. 224 of Dugundji [1] the conclusion follows.
Let $\gamma^+(x_0)$ be an orbit of the system (S) in ℓ_1 and $\Gamma^+(x_0) = i(\gamma^+(x_0))$. We are now ready to conclude the following theorem.

Theorem 4.6

$\Gamma^+(x_0)$ has a nonempty ω-limit set in ℓ_∞ if x_0 is in P.

Proof: $\gamma^+(x_0)$ is embedded in Q^+ by Lemma 4.1, and Q^+ is compact by Corollary 4.3. Thus, the theorem follows by Lemma 3 in Hale [1].

CHAPTER V

LIAPUNOV FUNCTIONS, LIMIT DYNAMICAL SYSTEMS AND THE STABILITY OF THE SYSTEM (S)

So far we have shown that the system (S) has a unique solution whose orbit lies in a bounded (but not necessarily a compact) subset in K. But, it was shown in Theorem 4.6 that the orbit has nonempty ω-limit set when embedded in ℓ_∞ by the inclusion map $i:\ell_1 \to \ell_\infty$. We use a Liapunov function, in the framework of Hale [1], to study the stability of the system (S). In the end of this chapter, we finally conclude that the solution of the system (S) approaches an equilibrium point for every initial point in ℓ_1.

§ 5.1 Dynamical Motion and the ω-limit Set

Let $y:R_+ X \ell_1 \to \ell_\infty$ be the motion defined by $y(t,x_0) = i \circ x(t, x_0)$. Then $||y(t,x_0)||_\infty \leq ||x(t,x_0)||_1$. By Theorem 4.6, the orbit $\Gamma^+(x_0) = \{z \text{ in } \ell_\infty : z = y(t,x_0),$ $t \geq 0$ has nonempty ω-limit set in ℓ_∞.

Lemma 5.1.1

Suppose $\{y^n\}$ is a sequence of points in ℓ_1 converging to a point y^0 in the topology of ℓ_∞. Then,

$$||y^0||_1 \leq \underline{\lim}_{n\to\infty} ||y^n||_1 .$$

Proof: Let (X,A,μ) be the measure space where X is the set of all integers, A the family of all subsets of X, and $\mu(E)$ the number of elements of E. The conclusion follows by applying Fatou's lemma (see Hewitt and Stromberg [1], pp. 172) to this measure space. That is, the inequality

$$\underline{\lim}_{n\to\infty}\, f_n d\mu \le \underline{\lim}_{n\to\infty}\, f_n d\mu$$

is translated into

$$\sum_i |y_i^0| \le \underline{\lim}_{n\to\infty} \sum_i |y_i^n| \ .$$

<u>Corollary 5.1.2</u>

If ϕ is a point in the ω-limit set of $y(t)$ in ℓ_∞, then $||\phi||_1 \le c$, where $c = ||x(0)||_1$.

Proof: Since $||y(t)||_1 = c$ by Theorem 4.1, for every increasing sequence $\{t_n\}$ such that $t_n\to\infty$ as $n\to\infty$, $||y(t_n)||_1 = c$ for every n. Therefore,

$$||\phi||_1 \le \underline{\lim}_{n\to\infty} ||y(t_n)||_1 = c$$

by Lemma 5.1.1.

The dynamical motion y, according to Theorem 4.6 and Corollary 5.1.2, gives information on the existence and the location of the ω-limit set of the dynamical system $x(t)$ when embedded in ℓ_∞. But invariance principle does not give any better result for a dynamical motion y. This is because, for the equality

$$\lim_{n\to\infty} y(t,x_n) = y(t,x_0)$$

to hold, we must have $x_n \to x_0$ in advance.

§ 5.2 The Invariance Principle

The following quotation from LaSalle's [1] statement of "Invariance Principle" will provide us with an intuitive idea of the principle: Invariance Principle: If the ω-limit set of a synamical system is positively invariant, then Liapunov function can be used to obtain information on the location of the ω-limit set.

To apply the invariance principle, we need to know that ω-limit set is nonempty. When a dynamical system arises from the solution of ordinary differential equations, a useful criteria to assure the existence of the ω-limit set is that the orbit belongs to a bounded set. But, if the dynamical system is defined on a Banach space which is not locally compact, the boundedness of the orbit is not enough to secure the existence of the ω-limit set (see Hale [1]).

Definition

Let X and Y be Banach spaces. A continuous map $f:X\to Y$ is called a compact operator if, for every bounded set E in X, f(E) is relatively compact in Y.

Hale [1] studied the invariance principle for the following situation: u is a dynamical system on Banach spaces X and Y. There is a compact linear injection $i:X\to Y$, and X is identified with its image in Y and one writes X Y.

But for our system (S) it is not clear whether a

bounded set in ℓ_1 belongs to a compact set in a larger Banach space. In fact, it is shown in Theorem 4.6 that every c_0-set, rather than a bounded set, in ℓ_1 belongs to a compact set in ℓ_∞. We demonstrate an invariance principle applied in this case. Our approach is done by using the concept of the "limit dynamical systems" introduced by Hale [1]. We introduce here some results of Hale's [1] about "limit dynamical systems" which are useful for discussing the asymptotic stability of the system (S). They are slightly modified here to serve our purpose. X and Y are Banach spaces throughout this chapter.

Definition

Suppose $X \subset Y$ and x is a dynamical system on X and Y. Let G be a subset of X. Let G* be the set consisting of the union of G and any ϕ in Y for which there is a ϕ in G such that ϕ belongs to $\omega_Y(\phi)$, the ω-limit set in Y of the orbit $\gamma^+(\phi)$ in Y; that is,

$$\omega_Y(\phi) = \bigcap_{\tau \geq 0} C\ell_Y \bigcup_{t \geq \tau} x(t,\phi) .$$

Then $x: R_+ \times G^* \to G^*$ is a dynamical system and we call x the limit dynamical system of G in Y.

Lemma 5.2.1

Suppose $X \subset Y$ and x is a dynamical system on X and Y. Let $G \subset X$. If ϕ is in G such that $\gamma^+(\phi)$ belongs to a compact set in Y, then the ω-limit set $\omega_Y(\phi)$ of the orbit through ϕ is nonempty, compact, connected set in G*, an invariant set of the limit dynamical system and $\text{dist}_Y(x(t,\phi),\omega_Y(\phi)) \to 0$

as $t \to \infty$.

This lemma is merely a restatement of lemma 3 of Hale [1] with B replaced by G^*.

§ 5.3 Liapunov Functions and Positively Invariant Sets

Lemma 5.3.1

The intersection of positively invariant sets is positively invariant.

Proof: Suppose M and N are positively invariant sets and $p \varepsilon M\ N$. Then $\gamma^+(p) \subset M$ and $\gamma^+(p)\ N$. Thus $\gamma^+(p) \subset M \cap N$.

Let x be a dynamical system on X. Suppose V is a continuous real valued function on X. Let

$$G^\beta := \{p \varepsilon X : V(p) < \beta\}.$$

Define the function $\dot{V}$ by

$$\dot{V}(p) = \overline{\lim_{t \to 0+}} \frac{1}{t}[V(x(t,p)) - V(p)].$$

Following LaSalle [1] and Hale [1], we say that a function $V : X \to R$ is a Liapunov function on a set G in X if 1) V is continuous on $\overline{G}$ and 2) $\dot{V}(p) \leq 0$ for all p in G. Let S be the set defined by

$$S = \{p \text{ in } \overline{G} : \dot{V}(p) = 0\}$$

and let N be the largest invariant set in S of the dynamical system.

Theorem 5.3.2

Suppose x is a dynamical system on X. Let G be a

subset of X. If V is a Liapunov function on G, such that each orbit starting in G belongs a compact subset of X, then $x(t,p) \to N$ as $t \to \infty$.

Proof: See Theorem 1 of Hale [1].

<u>Lemma 5.3.3</u>

If V is a Liapunov function on G, and H is a subset of G, then V is a Liapunov function on H.

Proof: Clear.

<u>Theorem 5.3.4</u>

Suppose M is a positively invariant set and V is a continuous function from X into R. Suppose that V is a Liapunov function on $M_\beta := M \cap G^\beta$. Then M_β is positively invariant.

Proof: Let ϕ be a point from $M_\beta = M \cap G^\beta$. Then $V(\phi) < \beta$ and $\dot{V}(\phi) \leq 0$. Therefore, for some number $\tau > 0$ if $t \in [0,\tau)$, then $V(x(t,\phi)) < \beta$ and $x(t,\phi) \in M_\beta$. Let $\tau_\phi = \sup \{\tau \in R^+ : x(t,\phi) \in M_\beta$ for all t in $[0,\tau)\}$. Suppose $\tau_\phi < \infty$. Since $x(t,\phi) \in M_\beta$ for $t \in [0,\tau_\phi)$ and V is a Liapunov function on M_β, i.e. $\dot{V}(x(t,\phi)) \leq 0$ for $t \in [0,\tau_\phi)$, $V(x(t,\phi))$ is a nondecreasing function in the interval $[0,\tau_\phi)$. Thus $V(x(t,\phi)) \leq V(\phi) < \beta$ for $t < \tau_\phi$. Since $x(\cdot,\phi)$ is continuous on R^+, $x(\tau_\phi,\phi)$ is in $\overline{M}_\beta$. Also, V being continuous on $\overline{\beta}$,

$$V(x(\tau_\phi,\phi)) \leq V(\phi) < \beta.$$

Thus $x(\tau_\phi,\phi) =: \psi$ is in β.

Repeating the same argument for ψ, there is a positive

number t' such that $x(s,\psi) \in M_\beta$ for $s \in [0,t']$. Since $x(s,\psi) = x(s,x(\tau_\phi,\phi)) = x(s + \tau_\phi, \phi)$, this is a contradiction to the maximality of τ_ϕ. Therefore, $\gamma^+(\phi)\ M_\beta$ for every ϕ in M_β. This completes the proof.

Corollary 5.3.5

If there is a Liapunov function on G^β, then G^β is a positively invariant set.

Proof: Let $M = X$, then $M_\beta = G^\beta$.

Remark. This Corollary is equivalent to the last part of Corollary 1 in LaSalle [1].

Theorem 5.3.6

Let $X \subset Y$ and y be a dynamical system on X and Y. Let G be a subset of X, such that each orbit starting in G belongs to a compact subset of Y. Let V_1 be a continuous function from X into R such that V_1 is a Liapunov function on $G_1 := \{\phi \text{ in } G : V_1(\phi) < n\}$. Let G_2 be a subset of Y. Let V_2 be a Liapunov function on G_2. Define $R := \{\phi \text{ in } \overline{G}_2 : \dot{V}_2(\phi) = 0\}$ and let N be the largest invariant set in R of the limit dynamical system. If $G_1 \subset G_2$ and ϕ is in G_1, then $y(t,\phi) \to N$ in Y as $t \to \infty$.

Proof: Since V_1 is a Liapunov function on G_1, it follows that $y(t,\phi)$ remains in G_1 for every $t \geq 0$ by Theorem 5.3.4. The hypothesis and Theorem 5.3.2 complete the proof.

§ 5.4 Stability of the System (S)

Suppose the system (S) has a unique solution in ℓ_∞. (A sufficient condition for the existence of a unique solution is given in Taylor [1], p. 220.) Let this solution

be $y = y(t,y_0)$. Notice that $y(t,y_0) = x(t,x_0)$ if $x_0 = y_0$ in K. Let $i : \ell_1 \to \ell_\infty$ be the inclusion map. Let $x : R_+ x K \to K$ be the dynamical system derived from solutions of the system (S). $K \subset \ell_1$. Then,

Lemma 5.4.1

If x_0 is in K and $\Gamma^+(x_0)$ is the positive orbit in ℓ_∞ through x_0, then $\omega(\Gamma^+(x_0))$ is a nonempty, compact, connected invariant set in ℓ_∞.

Proof: The result follows immediately from Theorem 4.6 and Lemma 5.2.1.

Let V be a sealar function on ℓ_∞ defined by $V(y) = -y_1^2$ for $y = (y_i)$ in ℓ_∞. Then 1° V is continuous, 2° $-c^2 \le V(y) \le 0$ for y in K and $||y||_1 = c$, and 3° $\dot{V}(y) = -2y_1\dot{y}_1 \le 0$ if y is in K. Therefore, V is a Liapunov function on K. Let R be the set defined by

$$R = \{y \text{ in } \overline{K} : \dot{V}(y) = 0\}$$

$$= \{y = (y_i) : \text{either } y_1 = 0 \text{ or } \dot{y}_1 = 0\}$$

$$= \{y = (y_i) : \text{either } y_1 = 0 \text{ or } 0 = y_2 = y_3 = \ldots.\}.$$

If $y = (y_i)$ is an element in R such that $y_1 = 0$ and $y_i > 0$ for some i, then $\dot{y}_1 > 0$. This implies that y(t) is not in R for some $t > 0$. Let

$$R_c = \{y = (y_i) \text{ in } K : 0 \le y_1 \le c,\ c = ||y||_1 \text{ and } y_i = 0 \text{ for every } i > 1\}.$$

Then a theorem follows immediately from Lemma 5.1 and

Theorem 5.4.2

Let x_0 be in K. Let $c = ||x_0||_1$. Then $x(t,x_0)$ approaches the set R_c as $t\to\infty$ in the ℓ_∞-norm.

Proof: Since every point in R_c is an equilibrium, R_c is the largest invariant set in R by Lemma 5.1.2. Then, $x(t,x_0)\to R_c$ as $t\to\infty$ by Theorem 5.3.2.

Lemma 5.4.3

Suppose that

1° x_0 is a point in K,

2° $c = ||x_0||_1$.

Then,

1* $x_1(t)$ converges to a number as $t\to\infty$.

2* $\lim_{t\to\infty} x_i(t) = 0$ for $i = 2,3,\ldots.$

3* $x(t,x_0)$ converges to the point $c^* = (c,0,0,\ldots)$ in the ℓ_∞-norm.

Proof: 1* By Lemma 5.1.2, $x_1(t) \le c$. Since K is a positively invariant set, $\dot{x}_1(t) \ge 0$.

2* By Theorem 5.4.2 $\sup_{i\ne 1} x_i(t)\to 0$ as $t\to\infty$. Therefore, $x_i(t)\to 0$ as $t\to\infty$ for every $i\ne 1$.

3* Let $0 < d < c$ and N be a positive integer such that

$$\sum_{i=1}^{N} x_i(0) > d.$$

Since

$$\frac{d}{dt}\sum_{i=1}^{N} x_i(t) = \dot{x}_1(t) + \dot{x}_2(t) + \ldots + \dot{x}_N(t)$$

$$= \sum_{i=1}^{\infty} (\text{positive number})\ x_{N+i}$$

then $\sum_{i=1}^{N} x_i(t)$ increases in t. Since $x_i(t)\to 0$ as $t\to\infty$ for $i\neq 1$ by 2* in this lemma, $\lim_{t\to\infty} x_1(t) > d$. Therefore, $\lim_{t\to\infty} x_1(t) = c$.

Corollary 5.4.4

If x_0 is in K and $c = ||x_0||_1$, then $x(t,x_0)$ converges to $c^* = (c,0,0,\ldots)$ in the ℓ_1-norm.

Proof: Since convergence in ℓ_∞-norm implies componentwise convergence and $||c^*||_1 = ||x(t,x_0)||$ for every $t \geq 0$, the conclusion follows by the theorem in p. 140 of S. Banach [1].

Definition

If G is positively invariant and each solution starting in G is relatively compact, then G is called strongly positively invariant.

Definition

Let H be a positively invariant set in G*. H is said to be uniformly asymptotically stable if for every x in G, $x(t,x_0)\to H$ as $t\to\infty$.

We list the following theorems without proof.

Theorem 5.4.5

K and P are strongly positively invariant sets in ℓ_∞.

Theorem 5.4.6

$S = \{x = (x_i) : x_i = 0 \text{ for } i \neq 1\}$ is uniformly

asymptotically stable for P and K.

5.5 Extension of the Stability of the System (S)

In this section, we generalize the results of the system (S) in section 5.4, to the whole space ℓ_1. If the initial condition $x_0 = (c_1, c_2, \dots c_i, \dots)$ is in ℓ_1, then one can write $x_0 = x_0^+ - x_0^-$ where both x_0^+ and x_0^- are in K. This is done by defining $x_0^+ = (a_1, a_2, \dots a_i, \dots)$ where

$$a_i = \frac{|c_i| + c_i}{2}$$

and $x_0^- = (b_1, b_2, \dots, b_i \dots)$ where

$$b_i = \frac{|c_i| - c_i}{2} \quad .$$

Let $x^+(t)$ and $x^-(t)$ be solutions of the system (S) such that $x^+(0) = x_0^+$ and $x^-(0) = x_0^-$. Then, $x(t) = x^+(t) - x^-(t)$ is the solution of the system (S) such that $x(0) = x_0$.

Definition

A system (S) in a Banach space X is said to have the stability property if the solution $x(t, x_0)$ of the system (S) approaches an equilibrium point in X as $t \to \infty$ in the norm of X.

Theorem 5.5.1

If the system (S) has a unique solution in ℓ_1, then (S) has the property of stability.

Proof: By Corollary 5.4.4, $x^+(t) \to (||x_0^+||_1, 0, 0, \dots)$ and $x^-(t) \to (||x_0^-||_1, 0, 0, \dots)$ in ℓ_1 -norm as $t \to \infty$. Therefore, $x(t)$ approaches the equilibrium point $c^* = (\sum_i c_i, 0, 0, \dots)$ as $t \to \infty$.

CHAPTER VI

BARBALAT'S LEMMA AND ASYMPTOTIC STABILITY

In this chapter we show that the solution of the system (S) starting from an initial condition K approaches the stationary point $(c,0,0,\dots)$ in ℓ_1-norm, where $c = ||x(0)||_1$. The main tool we use in this process is Barbalat's lemma. Otherwise, the argument is straightforward without applying any standard stability theory. In addition to the assumptions 1 and 2 in Chapter I concerning the m_{ij}'s and the α_i's, we assume in this chapter that $\sum_{j=2}^{\infty} m_{ij}$ is finite.

Let $V: K \to R_+$ be a function defined by

$$V(x) = \alpha x_1 + x_2 + \dots, \quad 0 < \alpha < 1.$$

Then, $\alpha ||x||_1 \le V(x) \le ||x||_1$.

This relationship is true on K but not true for the whole space ℓ_1.

<u>Lemma 6.1</u>

Let x_0 be in K, $c = ||x_0||_1$ and $x(t)$ be the solution of the system (S). Then

1° $x_1(t)$ converges to a finite number as $t \to \infty$.

2° $\lim_{t\to\infty} \dot{x}_1(t) = 0$

3° $\lim_{t\to\infty} x_i(t) = 0$ for $i = 2,3,\ldots.$

To prove this lemma we need the following lemma:

Barbalat's lemma (Popov [1], p. 211). Suppose f is a real-valued function uniformly continuous on R_+ such that

$$\lim_{t\to\infty} \int_0^t f(\tau)d\tau$$

exists and is finite. Then,

$$\lim_{t\to\infty} f(t) = 0.$$

Proof of Lemma 6.1:

1° Since $\sum x_i(t)$ is constant for all $t \geq 0$, we have $V(x(t)) = c - \beta x_1$, where $\alpha + \beta = 1$, and $\dot{V}(x(t)) = -\beta\dot{x}_1 \leq 0.$ Since $V(x(t)) \geq 0$ for $t \geq 0$, $V(x(t))$ converges to a limit as $t\to\infty$, and thus $x_1(t)$ converges to a limit as $t\to\infty$.

2° Let

$$g_n(x) = \sum_{j=2}^{n} m_{ij}x_j.$$

Then

$$\dot{g}_n(x) = \sum_{j=2}^{n} m_{ij}\dot{x}_j = -\sum_{j=2}^{n} m_{ij}\alpha_j x_j + \sum_{j=2}^{n} m_{ij}\sum_{k=j+1}^{\infty} m_{jk}x_k ,$$

and for $m < n$,

$$|\dot{g}_n - \dot{g}_m| \leq \sum_{j=m+1}^{\infty} m_{1j}\alpha_j x_j + \sum_{j=m+1}^{\infty} m_{1j}\sum_{k=j+1}^{\infty} m_{jk}x_k$$

$$\leq (A||x(t)||_1 + M||x(t)||_1)\sum_{j=m+1}^{\infty} m_{1j} .$$

Since $\sum_{j=2}^{\infty} m_{ij} < \infty$ and $||x(t)||_1$ is a constant $|\dot{g}_n - \dot{g}_m|\to 0$

as $n,m\to\infty$ regardless of t. On the other hand, if we write

$$g(x(t)) = \sum_{j=2}^{\infty} m_{1j}x_j(t), \quad \dot{g}(x(t)) = \sum_{j=2}^{\infty} m_{1j}\dot{x}_j \quad \text{and}$$

$$|\dot{g}(x(t))| \leq A\cdot M\cdot ||x(t)||_1 + M\sum_{n=2}^{\infty} m_{1n}\cdot ||x(t)||_1$$

$$< \text{Constant for all } t \geq 0.$$

Therefore, $\dot{x}_1(t) = g(x(t))$ is uniformly continuous.

Moreover,

$$\lim_{t\to\infty} \int_0^t \dot{x}_1(\tau)d\tau = x_1(\infty) - x_1(0) < \infty$$

by 1°. Barbalat's lemma implies that

$$\lim_{t\to\infty} \dot{x}_1(t) = 0$$

3° If $m_{1i} \neq 0$, then $m_{1i}x_i(t) \leq \sum_{j=2} m_{1j}x_j(t) = \dot{x}_1(t)$. This implies $x_i(t)\to 0$ as $t\to\infty$. In case $m_{1j} = 0$ for some positive integer $j > 1$, consider $\dot{x}_1 + \dot{x}_2 + \ldots + \dot{x}_{j-1}$ which also satisfies the conditions for Barbalat's lemma. Since the coefficient of the x_j-term in $\dot{x}_1 + \dot{x}_2 + \ldots + \dot{x}_{j-1}$ is $m_{1j} + m_{2j} + \ldots + m_{j-1j} = \alpha_j$ which is strictly positive, then we have

$$\alpha_j x_j(t) \leq \dot{x}_1 + \dot{x}_2 + \ldots + \dot{x}_{j-1}$$

and

$$\lim_{t\to\infty} x_j(t) = 0$$

Theorem 6.2

$s_1(t) := \sum_{i>1} x_i(t)$ converges to 0 as $t\to\infty$ and thus $x_1(t)\to c$ as $t\to\infty$.

Proof: Suppose $\lim_{t\to\infty} s_1(t) = s_1(\infty) > 0$, where the limit exists by Lemma 4.2. If ε is a positive number such that $\varepsilon < \frac{1}{2}s_1(\infty)$, then there is a positive integer N such that $s_n(0) < \varepsilon$ for $n > N$. Since $s_n(t)$ is monotonically decreasing by Lemma 4.2 $s_n(t) < \varepsilon$ for all $t \geq 0$ and $n > N$. Since $x_i(t)$ converges to 0 for every $i \neq 1$ as $t\to\infty$ by Lemma 6.1-3°, then there is a positive number T such that

$$\sum_{i=2}^{n} x_i(t) < \varepsilon$$

for $t \geq T$ for a fixed n. Therefore,

$$\begin{aligned}\lim_{t\to\infty} \sum_{i=1}^{\infty} x_i(t) &= \lim_{t\to\infty} x_1(t) + \lim_{t\to\infty} \sum_{i=2}^{n} x_i(t) + \lim_{t\to\infty} s_n(t) \\ &< c - s_1(\infty) + \varepsilon + \varepsilon \\ &= c - s_1(\infty) + 2\varepsilon < c.\end{aligned}$$

This contradicts the fact that $\sum_{i=1}^{\infty} x_i(t) = c$.

APPENDIX

APPROXIMATION OF SOLUTIONS TO AN INFINITE SYSTEM

If $A:\ell_1 \to \ell_g$, $1 \leq g < \infty$, is a bounded linear operator represented by an infinite matrix (a_{ij}), then the norm of A, which is defined by

$$||A||_g = \sup_j (\sum |a_{ij}|^g)^{\frac{1}{g}}$$

is finite (see Taylor [1], p. 220).

Let e_j be an element in ℓ_1 such that

$$e_j = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \delta_{ij} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad i = 1,2,\ldots$$

where δ_{ij} is the "Kronecker delta," which is defined by

$$\delta_{ij} = 0 \text{ if } i \neq j, \quad \delta_{ii} = 1 .$$

Theorem A.1

Suppose $\{B_n\}$ is a sequence of bounded linear operators from ℓ_1 into ℓ_1 and B_n is represented by an infinite matrix $(b_{ij}{}^n)$. If $\sup_n ||B_n x|| < \infty$ for every x in ℓ_1 and $\lim_{n\to\infty} ||B_n e_j|| = 0$ for every j, then $\lim_{n\to\infty} ||B_n x|| = 0$ for every x in ℓ_1.

Proof: Since $\sup_n ||B_n x|| < \infty$ for every x in ℓ_1, then, by the Banach-Steinhaus theorem, there is a positive constant K such that $||B_n|| \leq K$ for every n (see Rudin [2], p. 98). If $x = (x_i)$ is an element in ℓ_1, then we can write $x = \sum_j x_j e_j$. For every $\varepsilon > 0$, there is a positive integer $M(x,\varepsilon)$ such that $m > M$ implies $\sum_{j>m} |x_j| \leq \frac{\varepsilon}{2K}$.

Then by the interchangeability of double summation (see Rudin [1], p. 161),

$$\begin{aligned}
||B_n x|| &\leq \sum_i \sum_j |b_{ij}{}^n x_j| \\
&= \sum_j \sum_i |x_j||b_{ij}{}^n| \\
&= \sum_j |x_j| \sum_i |b_{ij}{}^n| \\
&\leq \sum_{j=1}^{m} |x_j| \sum_i |b_{ij}{}^n| + \frac{\varepsilon}{2K} \cdot K .
\end{aligned}$$

By the hypothesis, $||B_n e_j|| = \sum_i |b_{ij}{}^n| \to 0$ as $n \to \infty$ for every j. Then, there is a positive integer N such that $n > N$ implies

$$\sum_i |b_{ij}{}^n| < \frac{\varepsilon}{2||x||} \qquad \text{for } j = 1,2,\ldots m.$$

Therefore, whenever $n > N$, $||B_n x|| < \varepsilon$. This completes the proof.

Consider the infinite system of differential equations

$$\text{(T)} \qquad x = Ax,\ x(0) = c$$

where $x:R\to\ell_1$ is a function denoted by $x(t) = (x_i(t))$, $i = 1,2,\ldots$, $c = (c_i)$ is in ℓ_1 and $A = (a_{ij})$, $i,j = 1,2,\ldots$ is an infinite-dimensional constant matrix.

Let x_N be the solution of the approximate system

$$\dot{x} = A_N x,\ x(0) = c^N$$

where the infinite matrix A_N has the same Nth section as A and the rest of the entries are zeros. The initial condition $c^N = (c_i^N)$ is such that

$$c_i^N = \begin{cases} c_i \text{ if } i \le N \\ 0 \text{ if } i > N \end{cases} .$$

Definition

If $||x_N - x||\to 0$ uniformly as $n\to\infty$ on the interval $[0,t_0]$, then we say that the system (T) satisfies "Principe des Reduites."

The following theorem gives a sufficient condition for the system (S) given in Chapter I to satisfy "Principe des Reduites."

Theorem A.2

Suppose the matrix A for the system (T) is a triangular such that $a_{ij} = 0$ if $i > j$ and $\sup_j(\sum_i |a_{ij}|) < \infty$.

Then, the system (T) satisfies "Principe des Reduites" on any finite interval $[0,t_0]$.

Proof: Let x be the solution of the system (T) and x be the solution of the associated approximate system.

Let $B_0 = A$ and $B_N = A - A_N$, $N = 1,2,\ldots.$ Then,

$$\dot{x} - \dot{x}_N = Ax - A_N x_N = (A - A_N)x + A_N(x - x_N)$$

$$= B_N x + A_N(x - x_N).$$

By integration, we obtain

$$||x(t) - x_N(t)|| \le \int_0^t ||B_N x|| d_\tau + \int_0^t ||A_N||\ ||x - x_N|| d_\tau$$

Let

$$K_N = \int_0^{t_0} ||B_N x(\tau)|| d\tau.$$

Since $||A_N|| \le ||A||$ for every positive integer N, then

$$||x(t) - x_N(t)|| \le K_N + ||A|| \int_0^{t_0} ||x - x_N|| d\tau.$$

By Gronwall's inequality,

$$||x(t) - x_N(t)|| \le K_N \exp\left(\int_0^{t_0} ||A|| d\tau\right) \tag{A.1}$$

We will show next that $\lim_{N\to\infty} ||B_N x|| = 0$ for every x in ℓ_1:
Since $a_{iK} = 0$ if $i > K$ for the system (S), we have

$$\lim_{N\to\infty} ||B_N e_K|| = \lim_{N\to\infty} \sum_{i,j>N} |a_{ij}\delta_{jK}|$$

$$= \lim_{N\to\infty} \sum_{i>N} |a_{iK}| = 0 \text{ for each } k.$$

Also, $||B_N x|| \leq ||Ax|| \leq ||A||\ ||x|| < \infty$ for every x in ℓ_1. Thus, we have the above claim by Theorem 6.1.

If x(t) is the solution for the system (S), then $||B_N x(t)|| \leq ||Ax(t)||$ for every $t \geq 0$ and $||Ax(t)||$ is integrable on the finite interval $[0,t_0]$. Since

$\lim_{N\to\infty} ||B_N x(t)|| = 0$ for every t in $[0,t_0]$, then,

$$\lim_{N\to\infty} K_N = \lim_{N\to\infty} \int_0^{t_0} ||B_N x(t)||dt = \int_0^{t} \lim_{N\to\infty} ||B_N x(t)||dt = 0 \quad (A.2)$$

by the Lebesgue's Convergence Theorem. Equations (A1) and (A.2) together imply that x converges to x uniformly on any finite interval.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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